6 Change of basis and similarity

1. Let $\mathcal{B}$ and $\mathcal{B}'$ denote two given basis

$$\mathcal{B} = \left\{ \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\3\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix} \right\}, \qquad \mathcal{B}' = \left\{ \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\}$$

for vector space $\mathbb{R}^3$. Let u be a given vector which coordinates with respect to standard basis $\mathcal{S} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ are $(-2, 8, 6)$ (that is $[u]_{\mathcal{S}} = \begin{pmatrix}-2\\8\\-6\end{pmatrix}$). Find coordinates of vector u with respect to basis $\mathcal{B}$ (that is compute $[u]_{\mathcal{B}}$), and after that with the help of $[u]_{\mathcal{B}}$ compute $[u]_{\mathcal{B}'}$ (coordinates of vector u with respect to basis $\mathcal{B}'$).

2. Let $\mathcal{B} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ and $\mathcal{B}' = \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_n\}$ be bases for $\mathcal{V}$, and let $P = [I]_{\mathcal{B}\mathcal{B}'}$ where $I(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{V}$. Show that $[\boldsymbol{v}]_{\mathcal{B}'} = P[\boldsymbol{v}]_{\mathcal{B}}$ for all $\boldsymbol{v} \in \mathcal{V}$.

Changing Vector Coordinates Let $\mathcal{B} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ and $\mathcal{B}' = \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_n\}$ be bases for $\mathcal{V}$, and let T and P be the associated change of basis operator and change of basis matrix, respectively, i.e. $T(\boldsymbol{y}_i) = \boldsymbol{x}_i$ for each i, and

$$P = [T]_{\mathcal{B}} = [T]_{\mathcal{B}'} = [I]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} | & | & & | \\ [\boldsymbol{x}_1]_{\mathcal{B}'} & [\boldsymbol{x}_2]_{\mathcal{B}'} & ... & [\boldsymbol{x}_n]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}.$$

- $[\boldsymbol{v}]_{\mathcal{B}'} = P[\boldsymbol{v}]_{\mathcal{B}}$ for all $\boldsymbol{v} \in \mathcal{V}$.
- P is nonsingular.
- No other matrix can be used in place of $P = [I]_{\mathcal{B}\mathcal{B}'}$.

3. For the space $\mathcal{P}_2$ of polynomials of degree 2 or less, determine the change of basis matrix P from $\mathcal{B}$ to $\mathcal{B}'$, where

$$\mathcal{B} = \{1, t, t^2\} \quad \text{and} \quad \mathcal{B}' = \{1, 1+t, 1+t+t^2\},$$

and then find the coordinates of $q(t) = 3 + 2t + 4t^2$ relative to $\mathcal{B}'$.

Changing Matrix Coordinates Let A be a linear operator on $\mathcal{V}$, and let $\mathcal{B}$ and $\mathcal{B}'$ be two bases for $\mathcal{V}$. be two bases for $[A]_{\mathcal{B}}$ and $[A]_{\mathcal{B}'}$ are related as follows.

$$[A]_{\mathcal{B}} = P^{-1}[A]_{\mathcal{B}'}P, \qquad \text{where} \qquad P = [I]_{\mathcal{B}\mathcal{B}'}$$

is the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$. Equivalently,

$$[A]_{\mathcal{B}'} = Q^{-1}[A]_{\mathcal{B}}Q, \qquad \text{where} \qquad Q = [I]_{\mathcal{B}'\mathcal{B}} = P^{-1}$$

is the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$.

4. Consider the linear operator $A(x, y) = (y, -2x + 3y)$ on $\mathbb{R}^2$ along with the two bases

$$\mathcal{S} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} \quad \text{and} \quad \mathcal{S}' = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\}$$

First compute the coordinate matrix $[A]_{\mathcal{S}}$ as well as the change of basis matrix Q from $\mathcal{S}'$ to $\mathcal{S}$, and then use these two matrices to determine $[A]_{\mathcal{S}'}$.

5. Consider a matrix $M \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ to be a linear operator on $\mathbb{R}^n$ by defining $M(v) = Mv$ (matrix-vector multiplication). If $\mathcal{S}$ is the standard basis for $\mathbb{R}^n$, and if $\mathcal{S}' = \{q_1, q_2, ..., q_n\}$ is any other basis, describe $[M]_{\mathcal{S}}$ and $[M]_{\mathcal{S}'}$.

6. $A(x, y, z) = (x + 2y - z, -y, x + 7z)$ is a linear operator on $\mathbb{R}^3$. (a) Determine $[A]_{\mathcal{S}}$, where $\mathcal{S}$ is the standard basis. (b) Determine $[A]_{\mathcal{S}'}$ as well as the nonsingular matrix Q such that $[A]_{\mathcal{S}'} = Q^{-1}[A]_{\mathcal{S}}Q$ for $\mathcal{S}' = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$.

Similarity
- Matrices B, $C \in \mathrm{Mat}_{n\times n}(R)$ are said to be *similar matrices* whenever there exists a nonsingular matrix Q such that $B = Q^{-1}CQ$. We write $B \simeq C$ to denote that B and C are similar.
- The linear operator $f : \mathrm{Mat}_{n\times n}(\mathbb{R}) \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{R})$ defined by $f(C) = Q^{-1}CQ$ is called a *similarity transformation*.

7. The trace of a square matrix C is the sum of the diagonal entries

$$\operatorname{trace}(C) = \sum_i (C)_{ii}.$$

Show that trace is a similarity invariant, and explain why it makes sense to talk about the trace of a linear operator without regard to any particular basis. Then determine the trace of the linear operator on $\mathbb{R}^2$ that is defined by

$$A(x, y) = (y, -2x + 3y).$$

8. Show that two similar matrices must be coordinate matrices for the same linear operator.

Multiplication by a nonsingular matrix
Rank is invariant under multiplication by a nonsingular matrix. However, multiplication by rectangular or singular matrices can alter the rank.

9. Explain why rank is a similarity invariant.

10. Explain why similarity is transitive in the sense that $A \simeq B$ and $B \simeq C$ implies $A \simeq C$.

11. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \\ 0 & 1 & 5 \end{pmatrix}$ and $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$. Consider A as a linear operator on $\mathbb{R}^n$ by means of matrix multiplication $A(x) = Ax$. Determine $[A]_{\mathcal{B}}$.

12. Show that $A = \begin{pmatrix} 4 & 6 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & -3 \\ 6 & 10 \end{pmatrix}$ are similar matrices, and find a nonsingular matrix Q such that $C = Q^{-1}BQ$.

13. Let λ be a scalar such that $(C - \lambda I) \in \operatorname{Mat}_{n\times n}(\mathbb{R})$ is singular. (a) If $B \simeq C$, prove that $(B - \lambda I)$ is also singular. (b) Prove that $(B - \lambda_i I)$ is singular whenever $B \in \operatorname{Mat}_{n\times n}(\mathbb{R})$ is similar to

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

14. Let $\mathcal{B} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ and $\mathcal{B}' = \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_n\}$ be bases for $\mathcal{V}$, and let $P = [I]_{\mathcal{B}\mathcal{B}'}$ where $I(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{V}$. Define $T \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ by $T(y_i) = x_i$ for all i $(1 \le i \le n)$. Show that $P = [T]_{\mathcal{B}} = [T]_{\mathcal{B}'} = [I]_{\mathcal{B}\mathcal{B}'}$.

15. Let $\mathcal{B} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ and $\mathcal{B}' = \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_n\}$ be bases for $\mathcal{V}$, and let $P = [I]_{\mathcal{B}\mathcal{B}'}$ where $I(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{V}$. Show that P is nonsingular.

16. Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases for $\mathcal{V}$. Show that matrix P with the property that $[\boldsymbol{v}]_{\mathcal{B}'} = P[\boldsymbol{v}]_{\mathcal{B}}$ for all $\boldsymbol{v} \in \mathcal{V}$ is unique.

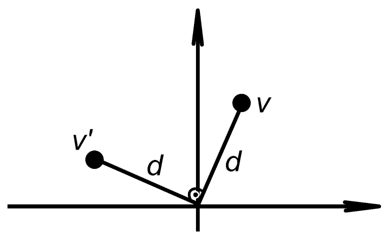
InC: 3, 4, 5, 7, 11, 12, 13. HW: 17, 18, 19 + several problems from the web page `http://osebje.famnit.upr.si/~penjic/linearnaAlgebra/`.

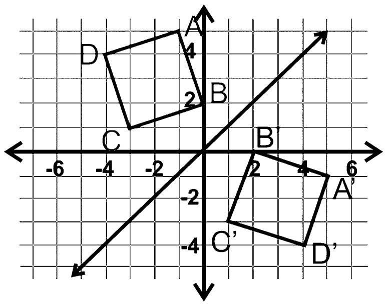
17. Let R_{90} denote rotation of 90° with centre of rotation in origin $(0, 0)$, so that point $v \in \mathbb{R}^2$ is mapped to point $v' \in \mathbb{R}^2$ (as is illustrated at figure right).

(a) Find coordinates of R_{90} with respect to standard basis.

(b) Determine what is rotation of point $v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for 90° about origin.

c) Find coordinates of R_{90} with respect to basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.





18. Let T denote linear operator on $\mathbb{R}^2$ which is reflection symmetry about line $y = x$ (for illustration what is reflection symmetry about line $y = x$ see $T(\square ABCD) = \square A'B'C'D'$ on figure left).

(a) Find coordinate matrix of T with respect to the standard basis.

(b) Compute $T(v)$, if we have that $v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

(c) Find coordinate matrix representation of T with respect to basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

19. Let T denote linear operator define on space $\mathbb{R}^2$ which first rotate vector for angle $\pi/3$ around origin in positive direction, and after that do reflection symmetry about line $y = x$. Find coordinate matrix representation of T with respect to basis $\mathcal{B} = \{(1, 1)^\top, (1, -1)^\top\}$ (in another words find $[T]_{\mathcal{B}}$). Find coordinates of vector $T(v)$ with respect to same basis $\mathcal{B}$, where v is arbitrary element from $\mathbb{R}^2$.